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ON THE MINIMUM NUMBER OF SPANNING TREES IN k-EDGE-CONNECTED GRAPHS

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ABSTRACT. We show that a k-edge-connected graph on n vertices has at least $n(k/2)^{n-1}$ spanning trees. This bound is tight if k is even and the extremal graph is the n-cycle with edge-multiplicities $k/2$. For k odd, however, there is a lower bound c_k^{n-1} where $c_k > k/2$. Specifically, $c_3 > 1.77$ and $c_5 > 2.75$. Not surprisingly, c_3 is smaller than the corresponding number for 4-edge-connected graphs. Examples show that $c_3 \leq \sqrt{2+\sqrt{3}} \approx 1.93$.

However, we have no examples of 5-edge-connected graphs with fewer spanning trees than the n-cycle with all edge-multiplicities (except one) equal to 3, which is almost 6-regular. We have no examples of 5-regular 5-edge-connected graphs with fewer than 3.09^{n-1} spanning trees which is more than the corresponding number for 6-regular 6-edge-connected graphs. The analogous surprising phenomenon occurs for each higher odd edge-connectivity and regularity.

1. Introduction

Every connected graph has a spanning tree, that is, a connected subgraph with no cycles containing all vertices of the graph. The number of spanning trees, denoted $\tau(G)$, is of importance in electrical networks, in particular, for expressing driving point resistances (effective resistances); see e.g. [9]. Kostochka [4] showed that, if G is a connected k-regular simple graph, then $k^{(1-O(\log k/k))} \leq \tau(G)^{1/n} \leq k$. But if we allow multiple edges, there are graphs with far less spanning trees. In this paper, we investigate the minimum number of spanning trees in k-edge-connected graphs with multiple edges. Since a loop is never contained in a spanning tree, we consider only graphs without loops.

In Section 2 we investigate how $\tau(G)$ changes when we replace a certain subgraph of G by another graph. In Section 3 we derive the lower bounds stated in the abstract. Since this bound is not tight for any odd edge-connectivity, we show in Section 4 that $\tau(G) \geq 1.774^{n-1}$ for every 3-edge-connected graph G on n vertices. The proof involves a new recursive description of the 3-connected cubic graphs; they can all be obtained from K_4 or $K_{3,3}$ by

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successively adding vertices or blowing vertices up to triangles. In Section 5, we consider the class of 5-regular 5-edge-connected graphs. Section 6 presents a class of k-regular k-edge-connected graphs which suggests that for odd $k > 3$, the minimum number of spanning trees might be obtained by an almost $(k+1)$-regular graph. Even more surprisingly, all examples of 5-regular, 5-edge-connected graphs with n vertices known to us have more than 3.09^{n-1} spanning trees while there are 6-regular, 6-edge-connected graphs with only $n3^{n-1}$ spanning trees.

We adopt the notation and terminology of Diestel [3]. We repeat a few important definitions. A **bridge** is an edge whose removal disconnects the graph. A graph is **k-edge-connected** if we need to remove at least k edges to disconnect the graph. A graph is **k-regular** if each vertex has k incident edges. A 3-regular graph is also called **cubic**. If e is an edge in a graph G, then G/e is the graph obtained by contracting e.

2. Lifting pairs of edges

Let G, H_1, H_2 be connected graphs, and $X \subseteq V(G)$, $X_i \subseteq V(H_i)$ for $i = 1, 2$ such that $|X| = |X_1| = |X_2|$. For $i = 1, 2$, let G_i be the graph obtained from $G \cup H_i$ by identifying X_i with X. We are interested in $\tau(G_1)/\tau(G_2)$. Let T be a spanning tree of G_1 or G_2. Then $T \cap G$ is a spanning forest of G. By comparing the number of ways of extending $T \cap G$ into a spanning tree of G_i using H_i, and taking the minimum ratio over all possible such forests, we can find a lower bound for $\tau(G_1)/\tau(G_2)$. Note that the number of ways of extending $T \cap G$ in G_i using H_i is exactly the number of spanning trees of the graph obtained from H_i by contracting each component of $T \cap G$ into a single vertex. This is made more precise in the following observation.

Observation 1. *Let G be a graph, and let $X \subseteq V(G)$ be a set of vertices. Suppose that G has two connected subgraphs G_0, G_1 such that $G_0 \cup G_1 = G$, $V(G_0 \cap G_1) = X$ and $E(G_0 \cap G_1) = \emptyset$. Let T be a spanning forest of G_0 such that each component contains at least one vertex in X. Then the number of ways of extending T to a spanning tree of G using edges in G_1 is $\tau(S_0)$, where S_0 is the graph obtained from $G_1 \cup T$ by contracting each component of T into a single vertex.*

Let $e = vu, f = vw$ be two adjacent edges of a graph. **Lifting** e, f means that we replace e, f by an edge uw if $u \neq w$. If $u = w$ we remove both edges e, f as we do not allow loops. By **lifting at** v we mean that we lift a pair of edges incident with v. A **complete lifting** at a vertex v with even degree is a sequence of liftings at v until no edges are left at v. Then we remove v.

For the following lemma, we define a constant c_d depending on a positive integer d:

$$c_d = \min_{d_1,d_2,\ldots,d_k} \min_H \frac{\prod_{i=1}^k d_i}{\tau(H)},$$

where the minimum is taken over all sequences of positive integers $d_1, d_2, \ldots, d_k$ with varying length k such that $\sum_{i=1}^k d_i = 2d$, and over all connected graphs H on k vertices with degree sequence $d_1', d_2', \ldots, d_k'$ such that $d_i' \le d_i$ for each i.

In the above definition of c_d, H has at most d edges, so $c_1 = 1$. Furthermore, $c_2 = 2$, $c_3 = 8/3$ and $c_4 = 18/5 = 3.6$, which are attained by a 2-cycle, a 3-cycle, and a 3-cycle plus an edge, respectively.

Lemma 1. *Let G be a graph with a vertex v of degree $2d$. Let G' be a graph obtained from G by a complete lifting at v. Then $\tau(G) \ge c_d \tau(G')$, where c_d is defined as above.*

Proof: Denote $G_0 = G - v$ and the neighbors of v in G by $v_1, v_2, \ldots, v_{2d}$, which are not necessarily distinct. We may assume that for each i, $v_{2i-1}v_{2i} \in E(G') \setminus E(G)$ resulting from lifting vv_{2i-1} and vv_{2i} unless $v_{2i-1} = v_{2i}$.

We consider a spanning forest, say T_0, of G_0 in which each component contains at least one of the neighbors of v. We shall estimate the number of ways of extending T_0 to a spanning tree using only edges not in G_0. The forest T_0 partitions the neighbors of v, say into $P_1, P_2, \ldots, P_k$ with sizes $|P_i| = d_i$, $\sum_{i=1}^k d_i = 2d$. By Observation 1, the number of ways of extending T_0 to a spanning tree of G (using no other edge of G_0) is precisely $\tau(S_0)$, where S_0 is the star graph at v with edge-multiplicities $d_1, d_2, \ldots, d_k$. Thus $\tau(S_0) = \prod_{i=1}^k d_i$. Likewise, the number of ways of extending T_0 to a spanning tree of G' is $\tau(S_0')$ where S_0' is the graph obtained from G' by contracting each component of T_0 into a single vertex, and then remove the remaining edges of G_0, if any. Let p_i be the vertex of S_0' corresponding to P_i. Then $\deg(p_i) \le d_i$, since each $v_j \in P_i$ provides p_i with at most one edge from $E(G') \setminus E(G_0)$. Therefore, the number of extensions of T_0 into spanning trees of G divided by the number of extensions to G' is at least $\min_H \prod_{i=1}^k d_i/\tau(H)$, where H is as described in the definition of c_d. Now we consider all possiblities for T_0 and get the inequality. □

Lemma 2. *Let G be a graph with a vertex v of degree $d \ge 3$. Let G' be a graph resulting from lifting edges vu, vw in G. Then $\tau(G) \ge (1 + \frac{4}{d^2-4})\tau(G')$.*

Proof: We consider a spanning forest, say T_0, of $G - v$ in which each component contains at least one of the neighbors of v. Then T_0 partitions the neighbors of v, say into $P_1, P_2, \ldots, P_k$ with sizes $|P_i| = d_i$, $\sum_{i=1}^k d_i = d$. By Observation 1, the number of ways to extend T_0 to a spanning tree of G is $\tau(S_0) = \prod_{i=1}^k d_i$, where S_0 is the star graph at v with edge-multiplicities

$d_1, d_2, \ldots, d_k$. Let S_0' be the graph obtained from G' by contracting each component of T_0 into a single vertex and then remove the remaining edges of $G-v$, if any. By Observation 1, there are $\tau(S_0')$ ways of extending T to a spanning tree of G'. If some P_j contains both u, w, then $\tau(S_0') = (d_j-2)\prod_{i\neq j} d_i$, so that either $\tau(S_0') = 0$ or $\tau(S_0)/\tau(S_0') = d_j/(d_j-2) > 1+4/(d^2-4)$.

If u, w are contained in two different parts, say P_i, P_j respectively, then S_0' is obtained from S_0 by lifting two edges connecting v to the two vertices corresponding to P_i and P_j. Thus,

$$\frac{\tau(S_0)}{\tau(S_0')} = \frac{d_i d_j}{d_i d_j - 1} \geq 1 + \frac{4}{d^2-4},$$

since $d_i + d_j \leq d$ which implies $d_i d_j \leq [(d_i+d_j)/2]^2 \leq d^2/4$.

By considering all possible such forests T_0, we get the inequality. □

3. k-edge-connected graphs

Let G be a connected graph with n vertices and m edges. Consider the pairs (e, T) where $e \in E(G)$ and T a spanning tree of G containing e. For each $e \in E(G)$ we have $\tau(G/e)$ such pairs and for each T, we have $n-1$ such pairs. Therefore $(n-1)\tau(G) = \sum_{e\in E(G)} \tau(G/e)$. Hence, G has an edge e such that $\tau(G/e)/\tau(G) \leq (n-1)/m$. We restate this conclusion as the following observation.

Observation 2. *Let G be a connected graph with $n > 1$ vertices and m edges. Then G has an edge e such that $\tau(G) \geq \frac{m}{n-1}\tau(G/e)$.*

Now we prove the first lower bound stated in the abstract.

Theorem 1. *Let G be a k-edge-connected graph on n vertices. Then G has at least $n(k/2)^{n-1}$ spanning trees. Moreover, G has more than $n(k/2)^{n-1}$ spanning trees unless k is even and G is a cycle whose edge-multiplicities are all $k/2$.*

Proof: We shall use induction on n. Since G is k-edge-connected, the minimum degree of G is at least k and thus $m \geq kn/2$. By Observation 2, G has an edge e such that $\tau(G) \geq \frac{m}{n-1}\tau(G/e) \geq \frac{kn}{2(n-1)}\tau(G/e)$. By the induction hypothesis, $\tau(G/e) \geq (n-1)(k/2)^{n-2}$ so that $\tau(G) \geq n(k/2)^{n-1}$. If equality holds, then k is even, $m = kn/2$, and G/e is a cycle where all edge-multiplicities are $k/2$. Moreover, any edge can play the role of e. This implies that all edge-multiplicities in G are $k/2$. If H denotes the subgraph of G obtained by replacing every multiple edge by a single edge, then H has the property that the contraction of any edge results in a cycle. Then also H is a cycle. □

For k even Theorem 1 is tight. However, for k odd we shall present a lower bound for the number of spanning trees in a k-edge-connected graph of the form c_k^{n-1} with $c_k > k/2$. For that, we shall use the following Theorem by Mader [6].

Theorem 2. *Let G be a connected graph on a vertex set $V \cup \{s\}$. If $\deg(s) \neq 3$ and s is not incident with bridges, then G has a lifting at s such that for each pair u, v of vertices in V, the maximum number of edge-disjoint paths between u, v does not decrease after the lifting.*

By Theorem 2 and Menger's Theorem, given a k-edge-connected graph and a vertex of degree $\geq k + 2$, we can find a lifting without decreasing the edge-connectivity. Thus by Lemma 2, the minimum number of spanning trees of a k-edge-connected graph on n vertices must be obtained by a graph whose degrees are only k or $k+1$. We state this as an observation for later use.

Observation 3. *If G is a k-edge-connected graph on n vertices with minimum $\tau(G)$, then each vertex of G has either k or $k + 1$ incident edges.*

Now we prove the following lower bound for odd edge-connectivity.

Theorem 3. *Let $k > 1$ be an odd number and let G be a k-edge-connected graph on n vertices. Then $\tau(G) \geq (kc_k/2)^{n-1}$, where $c_k = \sqrt{1 + \dfrac{4}{(k+3)^2 - 4}} > 1$*

Proof: Let e be an edge for which $\tau(G)/\tau(G/e)$ is maximum. By Observation 2 we know $\tau(G)/\tau(G/e) \geq k/2$. If the vertex of G/e resulting from the contraction of e, say v, has degree bigger than $k + 1$, then using Theorem 2 we can lift some pair of edges at v such that G/e after the lifting is still k-edge-connected. We do the lifting at v until the degree of v is at most $k+1$. Let H be the resulting graph. If $\tau(G)/\tau(H) \geq kc_k^2/2$ then we call e a *good* edge. Note that, if $H \neq G/e$, then by applying Lemma 2 at the last lifting, we see that e is good. Also, if e has multiplicity at least $(k+1)/2$, then $\tau(G)/\tau(H) \geq \tau(G)/\tau(G/e) \geq (k+1)/2 > kc_k^2/2$ so that e is good. If one of the ends of e has degree at least $k+1$, then either e has multiplicity at least $(k + 1)/2$, or the vertex obtained by the contraction of e has degree at least $k + 2$, so that e is good. Thus e is not good only if the ends of e both have degree precisely k. In particular, both ends of e have odd degree.

Now we repeat the contractions of an edge with maximum $\tau(G)/\tau(G/e)$, followed by liftings whenever possible, until only two vertices are left. Because of parity, among the $n-2$ contractions, at most $\lceil (n-2)/2 \rceil$ of them are edges whose ends both have odd degree. Thus at least $\lfloor (n-2)/2 \rfloor$ times we get an additional factor of c_k^2, so $\tau(G) \geq k \cdot (k/2)^{n-2} \cdot c_k^{2\lfloor (n-2)/2 \rfloor} > (kc_k/2)^{n-1}$. $\square$

By Theorem 3, Theorem 1 is not tight for any odd edge-connectivity, although it is tight for all even edge-connectivity. In the following we focus on k-edge-connected graphs where $k = 3, 5$.

4. 3-edge-connected graphs

Let G be a 3-edge-connected graph on n vertices. By Theorem 3, the lower bound $\tau(G) \geq n(3/2)^{n-1}$ is not tight. Kostochka [4] showed that a cubic simple 2-connected graph on n vertices has at least $8^{n/4} \approx 1.68^n$ spanning trees. This result is essentially best possible because of the cubic 2-connected graphs obtained by a collection of K_4's minus an edge by adding a matching. In this section, we prove the following theorem.

Theorem 4. *Let G be a 3-edge-connected graph on n vertices. Then $\tau(G) > 1.774^{n-1}$.*

Kreweras [5] showed that the prism graph on n vertices has approximately 1.93^n spanning trees; see Section 6. By Observation 3, a 3-edge-connected graph on n vertices with minimum number of spanning trees has vertex degrees only 3 and 4. Thus by Lemma 1, the following is enough to prove Theorem 4. Note that a cubic graph with more than two vertices has the same connectivity and edge-connectivity.

Theorem 5. *Let G be a 3-connected cubic graph on n vertices. Then $\tau(G) > 1.774^{n-1}$.*

An often used operation to construct a 3-connected cubic graph is to **join** two edges, i.e. for non-parallel edges e, f, we replace each edge by a path of length 2 and connect the two new vertices of degree 2 by an edge. Note that joining two non-parallel edges in a 3-connected cubic graph results in another 3-connected cubic graph. The following lemma explains how the number of spanning trees changes after joining.

Lemma 3. *Let G be a graph with two non-parallel edges e and f. Let G' be the graph obtained from G by joining e and f. Then $\tau(G') \geq (4 - r)\tau(G)$, where $r = \tau(G/e/f)/\tau(G) \leq 1$.*

Proof: We shall use Observation 1. We only consider the case when e, f are not adjacent, but the other case can be done likewise. Let $e = ab$ and $f = cd$. Let T be a spanning tree of G. Then $T - e - f$ is a spanning forest of G in which each component contains at least one of a, b, c and d. We shall consider how many ways $T - e - f$ can be extended to a spanning tree in G and G' respectively. For example, if $T - e - f$ has two components such that one of them contains a, c and the other contains b, d, then we can extend $T - e - f$ in two ways to a spanning tree of G, whereas there are eight ways for G'. In fact, there are at least four times as many extensions in G' as extensions in G, unless T contains both e and f, in which case we have a factor 3. Thus, $\tau(G') \geq 4(\tau(G) - \tau(G/e/f)) + 3\tau(G/e/f) = (4 - r)\tau(G)$. $\square$

To prove Theorem 5, we shall consider the following two operations to construct 3-connected cubic graphs.

(1) Let v be a vertex v in a graph such that $\deg(v) = 3$ and all three neighbors of v are distinct. Then the **blow-up** of v is obtained by joining two of the incident edges of v.

(2) Select three edges, which may not be pairwise distinct, but not all the same, and subdivide each of them so that we have three new vertices of degree 2. Add a new vertex v and an edge from v to each of the three vertices of degree 2. We call this a **vertex-addition**.

Since a blow-up is a join of two non-parallel edges, we get the following observation by Lemma 3.

Observation 4. *Let G be a graph with a vertex v of degree 3 whose neighbors are all distinct. Let G' be the graph obtained from G by a blow-up of v. Then $\tau(G') \geq 3\tau(G)$.*

Barnette and Grünbaum [1] and independently Titov [10] gave a characterization of 3-connected graphs which implies that every 3-connected cubic graph can be obtained from K_4 by successively joining edges. We shall here prove a stronger result for cubic graphs.

Theorem 6. *Let G be a 3-connected cubic graph with more than two vertices. Then G can be constructed from K_4 or $K_{3,3}$ by blow-ups and vertex-additions, such that blow-ups are never used consecutively.*

Proof: Our proof consists of two parts. We show that if G has no induced subgraph which is a subdivision of another 3-connected graph, then G is one of K_4, $K_{3,3}$ or the prism on 6 vertices defined in Section 6. Then we assume that G has a maximal induced subgraph, say H, which is a subdivision of another 3-connected graph H^*, and we show that G can be obtained from H^* by a vertex addition, possibly followed by a blow-up.

Suppose that G has no proper induced subgraph which is a subdivision of a 3-connected cubic graph. Let C be a cycle in G of minimum length so that C has no chord. Let v be a vertex in $G - V(C)$. Since G is 3-connected, Menger's Theorem implies that G has three paths P_1, P_2, P_3 where $P_i = vu_1^iu_2^i \dots u_{k_i}^iu_i$, $C \cap P_i = \{u_i\}$ for each i and the paths P_1, P_2, P_3 share only v. Let v be such a vertex with $k_1 + k_2 + k_3$ being smallest. Note that some k_i may be 0, implying that P_i is an edge. If G has an edge between the non-endvertices of two P_i's, say $u_i^1u_j^2$, then by taking $v = u_i^1$ instead and using $P_1 \cup P_3$ and $u_i^1u_j^2u_{j+1}^2 \dots u_{k_2}^2$, we get a smaller sum of the lengths of the paths unless u_j^2 is the neighbor of v in P_2. Similarly, we deduce that u_i^1 is also the neighbor of v in P_1. In this case, $vu_1^1u_1^2$ is a triangle and hence C

must also be a triangle, so that the vertex set of $C \cup P_1 \cup P_2 \cup P_3$, say V, induces a subgraph of G which is a subdivision of the prism graph. Thus by the assumption, G itself is the prism graph.

Hence we may assume that G has no edge between the non-endvertices of P_i's. Denote by $G[V]$ the subgraph of G induced by V. Suppose $k_1 \geq 1$ and some u_i^1 has a neighbor on C different from u_1. Because of the minimality of $k_1 + k_2 + k_3$, we have $i = k_1$ and by taking $v = u_{k_1}^1$ and using its two neighbors on C, we see $k_2 = k_3 = 0$. Therefore $G[V]$ is a subdivision of either the prism graph or $K_{3,3}$, so that again G itself is either the prism graph or $K_{3,3}$. The remaining case leaves no other edge in $G[V]$ than $C \cup P_1 \cup P_2 \cup P_3$, which is a subdivision of K_4. Thus in this case G itself is K_4. This completes the first part.

Now we assume that G has an induced proper subgraph which is a subdivision of a 3-connected cubic graph. Let H be a maximal such subgraph. Let us call a path in H *suspended* if its ends both have degree 3 in H and all other vertices in the path have degree 2 in H. Suspended paths intersect only at their ends. By replacing each suspended path of H by an edge between its ends, we get a 3-connected cubic graph, which we denote H^*. Since G is 3-connected, H has at least two suspended paths. If G has a vertex, say v, outside H which has neighbors in at least two distinct suspended paths of H, then the subgraph of G induced by $V(H) \cup \{v\}$ is a subdivision of a 3-connected graph, which must be G because of the maximality of H. Then G can be obtained from H^* by the vertex-addition of v. Thus we may assume that for each vertex in $V(G) \setminus V(H)$, its neighbors in H, if any, are in a single suspended path of H. Also, we may assume that $|V(G) \setminus V(H)| > 1$. If $V(G) \setminus V(H) = \{u, v\}$, then u and v are adjacent, and they have neighbors in distinct suspended paths. Thus we can obtain G from H^* by first vertex-adding u and then a blow-up to make v. Therefore, we assume that $|V(G) \setminus V(H)| > 2$.

Since G is 3-connected, at least one component of $G - V(H)$ has edges to two distinct suspended paths of H. Thus G has a path of length > 1 between distinct suspended paths of H which intersects H at only its ends. Let $P = v_0 v_1 \ldots v_k$ be such a path with smallest length. Since P has no chord, the subgraph of G induced by $H \cup P$ is a subdivision of a 3-connected graph, so that $V(H) \cup V(P) = V(G)$, implying $k \geq 4$. By assumption, the neighbors of v_1 and v_{k-1}, respectively, are in different suspended paths of H. Let v be the neighbor of v_2 in H. Then either $v_0 v_1 v_2 v$ or $v v_2 v_3 \ldots v_k$ contradicts the minimality of P, a contradiction which completes the proof. $\square$

Let c be the positive real solution of the equation $x^4 - 3x^2 - 1 = 0$ which is approximately $c \approx 1.8174$. Note that a vertex-addition is equivalent to a joining of two edges and then joining the new edge with an edge.

Lemma 4. *Let G_0 be a 3-connected graph and let G be a graph obtained from G_0 by joining two non-parallel edges of G_0, where e denotes the joining edge. Let G' be a graph obtained from G by joining e with another edge f. Then either $\tau(G') \geq c^2\tau(G)$ or $\tau(G') \geq c^4\tau(G_0)$.*

Proof: Let $r = \tau(G/e/f)/\tau(G)$ be as in Lemma 3. Let $r' = \tau(G/e)/\tau(G)$ so that $\tau(G)/\tau(G-e) = 1/(1-r')$. Since $r' \geq r$, Lemma 3 implies $\tau(G') \geq (4-r)\tau(G) \geq (4-r')\tau(G)$. If $4-r' \geq c^2$ then we are done. Thus we may assume that $4-r' < c^2$, equivalently $1-r' < c^2-3$. By modifying the equation for c, we get $1 + 3/(c^2-3) = c^4$, so that

$$\begin{aligned}\tau(G') &\geq (4-r')\tau(G) = \frac{(4-r')\tau(G)}{\tau(G_0)}\tau(G_0) \geq \frac{(4-r')\tau(G)}{\tau(G-e)}\tau(G_0) = \frac{4-r'}{1-r'}\tau(G_0) \\ &= \left(1 + \frac{3}{1-r'}\right)\tau(G_0) > \left(1 + \frac{3}{c^2-3}\right)\tau(G_0) = c^4\tau(G_0).\end{aligned}$$

□

Proof of Theorem 5: We shall prove $\tau(G) \geq (3c^2)^{(n-1)/4}$ by induction on $n = |V(G)|$, where c is the constant used in Lemma 4. We may assume that $n \geq 8$ because K_4, $K_{3,3}$ and the prism on 6 vertices have 16, 81 and 75 spanning trees, respectively. By Theorem 6, G can be obtained from K_4 or $K_{3,3}$ by repeatedly applying vertex-additions and blow-ups. If the last operation is a vertex-addition, then by Lemma 4, $\tau(G) \geq c^2\tau(G')$ or $\tau(G) \geq c^4\tau(G'')$ for some 3-connected cubic graph G' with $n-2$ vertices or G'' with $n-4$ vertices, so we are done. Otherwise, G can be obtained from a 3-connected cubic graph using a vertex-addition and then a blow-up. By Observation 4, a blow-up multiplies the number of spanning trees by at least 3, so that using Lemma 4, $\tau(G) \geq 3c^2\tau(G')$ or $\tau(G) \geq 3c^4\tau(G'')$ for some 3-edge-connected cubic graph G' with $n-4$ vertices or G'' with $n-6$ vertices. By the induction hypothesis, $\tau(G) \geq (3c^2)^{(n-1)/4} > 1.774^{n-1}$. □

5. 5-regular 5-edge-connected graphs

Let G be a 5-regular 5-edge-connected graph. A **5-cut** is a set of edges E with $|E| = 5$ such that $G - E$ is disconnected. If one of the components of $G - E$ is a single vertex, then we call E **trivial**. Otherwise we call E **nontrivial**. A **5-side** is a set $X \subseteq V(G)$ such that $\delta(X)$(that is, the set of edges with precisely one end in X) is a nontrivial 5-cut. If a 5-side X has the property that no nontrivial 5-cut contains an edge with both ends in X, then X is called **minimal**.

Lemma 5. *Let G be a 5-regular 5-edge-connected graph. If G has a nontrivial 5-cut, then G has a minimal 5-side.*

Proof: Let A be a 5-side which is not minimal. Then some nontrivial 5-cut $S = \delta(B)$ contains an edge uv with $u \in A \cap B$ and $v \in A \cap B^c$. Let $T = \delta(A)$. One of the sets $A \cap B$, $A \cap B^c$, $A^c \cap B$ or $A^c \cap B^c$ is empty because G is 5-edge-connected, S, T are 5-cuts and 5 is odd. Since $u \in A \cap B$ and $v \in A \cap B^c$, either $A^c \cap B$ or $A^c \cap B^c$ is empty, so that either $A \cap B$ or $A \cap B^c$ is a 5-side strictly smaller than A. If it is not minimal, then we repeat the argument until we eventually find a minimal 5-side. $\square$

Lemma 6. *Let G be a connected graph with a connected subgraph H. If G' is the graph obtained by contracting H into a single vertex, then $\tau(G) \geq \tau(H)\tau(G')$.*

Proof: For each pair S, T of spanning trees of H, G', we can expand the contracted vertex of G' using S to get a spanning tree of G. $\square$

Theorem 7. *Let G be a 5-regular 5-edge-connected graph on n vertices. Then $\tau(G) \geq 7.6^{(n-1)/2} \approx 2.7568^{n-1}$.*

Proof: We shall use induction on n. Being 5-regular and 5-edge-connected, G has no edge of multiplicity at least 3. If G has a nontrivial 5-cut, then by Lemma 5, we can find a minimal 5-side, and we let $e = uv$ be an edge inside that minimal side. Otherwise let $e = uv$ be an arbitrary edge.

Suppose first e has multiplicity 1. G/e has a vertex of degree 8, which we can completely lift using Theorem 2. Denote the resulting 5-regular 5-edge-connected graph by G'. By Lemma 1, $\tau(G/e) \geq 3.6\tau(G')$. Now we consider $G - e$. Since e is not contained in any nontrivial 5-cut, $G - e$ has at least 5 edge-disjoint paths between any pair of vertices distinct from the ends of e. Thus by Theorem 2, we can completely lift u, v in $G - e$ so that the resulting graph, say G'', is 5-edge-connected and 5-regular. By Lemma 1, $\tau(G - e) \geq 4\tau(G'')$ and by the induction hypothesis,

$$\tau(G) = \tau(G/e) + \tau(G - e) \geq 3.6\tau(G') + 4\tau(G'') \geq 7.6^{(n-1)/2}.$$

Now we may assume that every edge of G with multiplicity 1 is contained in a nontrivial 5-cut. Let X be a minimal 5-side. Since the edges inside X are not contained in any nontrivial 5-cut, every edge inside X must be a double edge. Hence every vertex in X is incident with $\delta(X)$, so that X is the 5-double-cycle which has 80 spanning trees. By Lemma 6, $\tau(G) \geq 80\tau(G/X)$, and by the induction hypothesis, $\tau(G) \geq 7.6^{(n-1)/2}$. $\square$

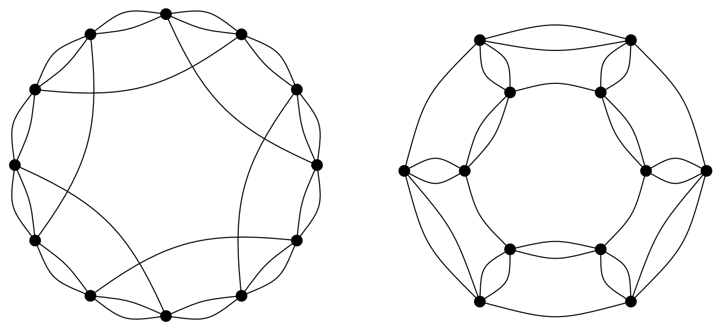

FIGURE 1. Two different drawings of $MP_{12}(5)$

6. EXAMPLES OF k-REGULAR k-EDGE-CONNECTED GRAPHS WITH FEW SPANNING TREES

In this section, we describe some k-regular k-edge-connected graphs for odd k, leading to a conjecture that the minimum number of spanning trees of a k-edge-connected graph is obtained by a nearly $(k+1)$-regular graph if k is odd. See Open Problems 2, 3 in Section 7.

Let kC_n be the cycle of length n whose edge multiplicities are all k. By Theorem 1, when k is even, $\frac{k}{2}C_n$ has the minimum number of spanning trees among all k-edge-connected graphs on n vertices. If k is odd, $\frac{k+1}{2}C_n$ minus an edge, say $\frac{k+1}{2}C_n - e$, gives an upper bound on the minimum number of spanning trees of a k-edge-connected graph on n vertices. The spanning trees of $\frac{k+1}{2}C_n - e$ belong to either the unique path with uniform edge-multiplicity $\frac{k+1}{2}$ or the $(n-1)$ paths in which the edge-multiplicities are $\frac{k+1}{2}$ except an edge with one less multiplicity. Thus, the number of spanning trees of $\frac{k+1}{2}C_n - e$ is

$$\left(\frac{k+1}{2}\right)^{n-1} + (n-1)\left(\frac{k+1}{2}\right)^{n-2}\frac{k-1}{2} = \left(1 + (n-1)\frac{k-1}{k+1}\right)\left(\frac{k+1}{2}\right)^{n-1}.$$

We conjecture that this number is the minimum number of spanning trees of a k-edge-connected graph on n vertices when k is an odd number bigger than 3, and $\frac{k+1}{2}C_n - e$ is the unique extremal graph realizing the number.

We do not know any k-regular k-edge-connected graphs with that few spanning trees. Instead, there are k-regular k-edge-connected graphs with $(\frac{k+2}{2} + O(\frac{1}{k}))^{n-1}$ spanning trees, namely multiprisms defined below.

The **prism** P_{2n} is the Cartesian product of C_n and K_2. If $n > 2$ is a natural number and k is odd then the **multiprism** $MP_{2n}(k)$ is defined as follows:

(1) Let $v_0, v_1, \ldots, v_{2n-1}$ be the vertices of $\frac{k-1}{2}C_{2n}$, where v_i and v_{i+1} are adjacent for all i.

(2) Add edges v_0v_3, v_2v_5, $\ldots$, $v_{2n-4}v_{2n-1}$ and $v_{2n-2}v_1$.

If n is even, $MP_{2n}(k)$ can also be obtained by choosing a Hamilton cycle of P_{2n} and replace its edges by $(k-1)/2$-multiple edges. See Figure 1.

Kreweras [5] determined the exact number of spanning trees in the prisms. Rubey [8, p. 40] showed another method, which can be used to give the exact formula for $\tau(MP_{2n}(k))$; c.f. [7]. Let $k = 2s+1$. Then

$$\begin{aligned}\tau(MP_{2n}(2s+1)) = \frac{sn}{A-B}A^n &\left[1 + 2\frac{s^2A^{n-2} - s^n}{A^n - s^2A^{n-2}} + \frac{1+s^2}{A}\frac{A^n - s^n}{A^n - s^2A^{n-2}}\right] \\ - B^n &\left[1 + 2\frac{s^2B^{n-2} - s^n}{B^n - s^2B^{n-2}} + \frac{1+s^2}{B}\frac{B^n - s^n}{B^n - s^2B^{n-2}}\right],\end{aligned}$$

where $A = \frac{s}{2}\left(s+3+\sqrt{s^2+6s+5}\right)$ and $B = \frac{s}{2}\left(s+3-\sqrt{s^2+6s+5}\right)$.

Thus $\lim_{n\to\infty} \tau(MP_{2n}(k))^{1/2n} = A^{1/2} = s + \frac{3}{2} + O(\frac{1}{s}) = \frac{k+2}{2} + O(\frac{1}{k})$.

In particular, $\tau(MP_n(5)) > 3.09^n$ for large even n.

Note again that the number of spanning trees of $MP_{2n}(k)$, which is k-regular k-edge-connected, is asymptotically $\left(\frac{k+2}{2}\right)^{2n}$. As we have a $(k+1)$-regular $(k+1)$-edge-connected graph, namely $\frac{k+1}{2}C_{2n}$, with asymptotically less spanning trees, we suspect that the minimum number of spanning trees of a k-edge-connected graph, when k is odd, may be achieved by an almost $(k+1)$-regular graph. Specifically, we believe that for every odd $k \geq 5$, $\frac{k+1}{2}C_n$ minus an edge has the fewest spanning trees among all k-edge-connected graphs on n vertices.

7. OPEN PROBLEMS

For $\mathcal{C}$ an infinite class of finite graphs, define $c(\mathcal{C}) = \liminf\{\tau(G)^{1/n} : G \in \mathcal{C}, n = |V(G)|\}$. Let $\mathcal{C}_k$ be the class of k-edge-connected graphs. Let $\mathcal{C}'_k$ be the class of k-regular k-edge-connected graphs. We have proved that $c(\mathcal{C}_k) = c(\mathcal{C}'_k) = k/2$ for k even and that $k/2 < c(\mathcal{C}_k) \leq c(\mathcal{C}'_k)$ for k odd. Moreover $1.774 < c(\mathcal{C}_3) = c(\mathcal{C}'_3) \leq 1.932$, $2.75 < c(\mathcal{C}_5) \leq 3$ and $c(\mathcal{C}_5) \leq c(\mathcal{C}'_5) < 3.1$.

Open Problem 1. *Is $c(\mathcal{C}_3) = \sqrt{2+\sqrt{3}} \approx 1.93$, which is obtained by the prisms?*

Open Problem 2. *Is $c(\mathcal{C}_k) = c(\mathcal{C}_{k+1}) = \frac{k+1}{2}$ for k odd, $k \geq 5$?*

Open Problem 3. *Is $c(\mathcal{C}'_k) = k/2 + 1 + O(1/k)$ for k odd?*

Open Problem 4. *Is $c(\mathcal{C}'_5) = \sqrt{5+\sqrt{21}} \approx 3.0956$, which is obtained by the multiprisms $MP_n(5)$?*

Even if Problems 2 and 3 both have negative answers, we may still ask if $c(\mathcal{C}'_k) > c(\mathcal{C}_{k+1})$ for each odd $k \geq 5$.

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